TASI Lecture 4: Differential Cohomology

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1. Outline

2. The Group Of Cheeger-Simons Characters

There is a natural generalization of the coupling $\exp i\oint_\gamma qe A$ to that for electric branes for a generalized Abelian gauge field.

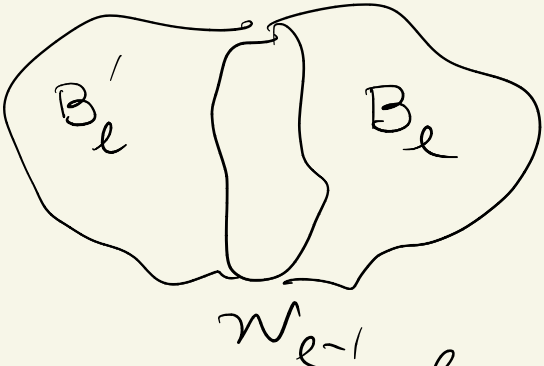
Definition A C-S character of degree ℓ is a homomorphism

$$\chi \in \mathrm{Hom}\left(Z_{\ell-1}(M), U(1)\right)$$

s.t. $\exists\, F \in \Omega^\ell(M)$ s.t. if $W_{\ell-1} = \partial B_\ell$

Then $\chi(W_{\ell-1}) = \exp\left(i \int_{B_\ell} F\right)$

<u>Remark</u>: F is called "the field strength" of the character χ. The same argument



$$\Longrightarrow \quad F \in \Omega^{\ell}_{\mathbb{Z}'}(M) \subset \Omega^{\ell}_{d}(M)$$

$$\mathbb{Z}' := 2\pi\mathbb{Z}$$

The Abelian group of all CS characters is denoted $\check{H}^{\ell}(M)$. It is also known as a <u>differential cohomology group</u>.

If $A \in \Omega^{\ell-1}(M)$ is globally well-defined then we can define $\chi_A(\mathcal{W}_{\ell-1}) := \exp\left(i \int_{\mathcal{W}_{\ell-1}} A \right)$

Then $F = dA$, and all the periods vanish.

Note that if $\Lambda \in \Omega^{\ell-2}(M)$ then A and $A + d\Lambda$ define the same character. More generally if $\omega \in \Omega^{\ell-1}_{\mathbb{Z}}(M)$ then A and $A+\omega$ define the same character.

Topologically trivial: $\Omega^{\ell-1} / \Omega^{\ell-1}_{\mathbb{Z}}$

But we can have topologically nontrivial characters. In general F can have nonzero periods.

This is looking a lot like gauge theory of a $(\ell-1)$ gauge potential. But there is no convenient model in terms of bundles and connections for higher form gauge potentials.

So we make a physical proposal that the proper way to describe the gauge invariant information in generalized Maxwell theory is:

$$\check{H}^{\ell}(M) = \left\{ \begin{array}{l} \text{gauge equivalence} \\ \text{classes of generalized} \\ \text{Maxwell fields with} \\ \ell\text{-form fieldstrength} \\ \text{on } M \end{array} \right\}$$

Two important examples:

1. $\check{H}^1(M) \cong$ Functions$\left(M \to \mathbb{R}/\mathbb{Z}\right)$

2. $\check{H}^2(M) \cong \left\{ \begin{array}{l} \text{gauge equiv. classes} \\ \text{of } (P,\nabla) \longrightarrow M \\ P = \text{principal } U(1) \text{ bundle} \\ \nabla = \text{connection} \end{array} \right.$

Remark: The action of a p_e-brane electrically charged for a generalized Maxwell field $F \in \Omega^l$ (hence $p_e = l-2$) has a world volume action

$$\exp\left(i \int_{W_e} A\right)$$

in topologically trivial backgnds

If we declare that the proper generalization of ℓ-form generalized Maxwell theory to topologically nontrivial field config's entails the identification of $\check{H}^{\ell}(M)$ with the set of gauge equivalence classes then it is natural to say that the coupling $\exp\left(i \int_{W_{\ell-1}} A\right)$ of an electrically charged brane in such a background is precisely the character $\chi(W_{\ell-1})$.

Remark on group structure:

$$Z_k(M) = \ker \partial : C_k \longrightarrow C_{k-1}$$

is a subgroup of the Abelian group C_k (= free Abelian group generated by continuous maps $\phi : \Delta^k \to M$)

Also $\mathrm{Hom}(Z_{\ell-1}(M), U(1))$ comes ~~a~~ Abelian group structure.

Latter we'll put a nontrivial ring structure on $\bigoplus_{\ell} \check{H}^{\ell}(M)$ so it's good to work with $\mathrm{Hom}(Z_{\ell-1}(M), \mathbb{R}/\mathbb{Z}')$ with

$$(\chi_1 + \chi_2)(W) := \chi_1(W) + \chi_2(W) \mod \mathbb{Z}'$$

Exercise: Show that if we repeat the definition of a Cheeger-Simons character but for $\mathrm{Hom}(Z_{\ell-1}(M), \mathbb{R})$ then the fieldstrength must have zero periods.

3. Properties Of $\check{H}^\ell(M)$: The Dancing Hexagon.

We now analyze the structure of $\check{H}^\ell(M)$ as an Abelian group through a number of interlocked exact sequences.

Before we get going we need a few math preliminaries. (See texts on group theory and algebraic topology for proofs.)

1. Abelian groups have a canonically defined subgroup:

$$\mathrm{Tors}(A) = \{ a \in A \mid \exists n \in \mathbb{Z} \text{ s.t. } na = 0 \}$$

2. If A is a <u>finitely generated</u> Abelian group then $\mathrm{Tors}(A)$ is a finite Abelian group and:

$$0 \to \mathrm{Tors}(A) \xrightarrow{i} A \xrightarrow{\pi} \bar{A} \to 0$$

$\bar{A} \cong \mathbb{Z}^b \quad b =$ "rank of A" and

$$A \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^b$$

3. If A is a <u>compact</u> topological Abelian group then

$$A_0 = \text{Connected cpt of identity} \cong U(1)^r$$

and

$$1 \longrightarrow A_0 \longrightarrow A \longrightarrow \underbrace{\pi_0(A)}_{\text{finite Abelian group}} \longrightarrow 1$$

So as a topological space A is a disjoint union of $|\pi_0(A)|$ copies of an r-dim'l torus but the group structure in general is <u>not</u> a direct product: The sequence does not split.

4. If $\cdots \leftarrow C_k \xleftarrow{\partial} C_{k+1} \xleftarrow{\partial} \cdots$

is a chain complex of Abelian groups with $\partial^2 = 0$

and $\cdots \longrightarrow C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \cdots$

is the dual cochain complex with

$C^k = \mathrm{Hom}(C_k, \mathbb{Z})$ Then (co)homology

with coefficients

$H^\bullet(C^\bullet; A)$ is obtained by $\otimes A$.

the relation to $H^\bullet(C^\bullet)$ is

subtle:

$$0 \to \mathrm{Ext}(H_{j-1}(C_\bullet), A) \to H^j(C^\bullet, A)$$

$$\longrightarrow \mathrm{Hom}(H_j(C_\bullet), A) \to 0$$

5. For a compact manifold $H^k(M,\mathbb{Z})$ is a finitely generated Abelian group and

$$H^k(M, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(H_k(M,\mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

is a compact Abelian group

6. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

induces a LES of cohomology

In general, if we have a differential Abelian group (B,d) and a differential Abelian subgroup (A,d) then we have:

$$0 \to A \xrightarrow{i} B \to B/A \to 0$$

So $d^2 = 0$ on A, B and

$d(i(A)) \subset i(A)$ then there

is a degree 1 map

$$\delta : H(B/A) \longrightarrow H(A)$$

$$\delta : [b+A] \longmapsto [db]$$

This makes sense: If $d_{B/A}(b+A) = 0$ in B/A it means $db \in A$. But it could be that $db \in A$ is nonzero. It is certainly in the kernel of d, but it might not be in the image of d restricted to A.

So $[db] \neq 0$ is possible.

Let us apply this to

$A = C^\bullet(M) \quad B = C^\bullet(M)\otimes\mathbb{R}$

$B/A = C^\bullet(M)\otimes\mathbb{R}/\mathbb{Z}$

$$\begin{tikzcd}
0 \arrow[r] & C^{k+1}(M,\mathbb{Z}) \arrow[r] & C^{k+1}(M,\mathbb{R}) \arrow[r, "\pi"] & C^{k+1}(M,\mathbb{R}/\mathbb{Z}) \arrow[r] & {} \\
0 \arrow[r] & C^{k}(M,\mathbb{Z}) \arrow[r] \arrow[u, "d"] & C^{k}(M,\mathbb{R}) \arrow[r] \arrow[u, "d"] & C^{k}(M,\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[u, "d"] & 0 \\
 & C^{k-1}(M,\mathbb{Z}) \arrow[u, "d"] & {} \arrow[u] & C^{k-1}(M,\mathbb{R}/\mathbb{Z}) \arrow[u, "d"] &
\end{tikzcd}$$

For $\bar{a}\in Z^k(M,\mathbb{R}/\mathbb{Z})$ lift to $a\in C^k(M,\mathbb{R})$

$\pi(da)=0$ so da has a lift to $b\in C^{k+1}(M,\mathbb{Z})$

but $db=0$ so $b\in Z^{k+1}(M,\mathbb{R}/\mathbb{Z})$

With a little thought one sees that we have a LES

$$\cdots \to H^k(M,\mathbb{Z}) \xrightarrow{i} H^k(M,\mathbb{R}) \xrightarrow{\psi} H^k(M,\mathbb{R}/\mathbb{Z}) \to H^{k+1}(M,\mathbb{Z}) \xrightarrow{i} H^{k+1}(M,\mathbb{R}) \longrightarrow \cdots$$

β is called the Bockstein map.

Since the sequence is exact

$$\operatorname{im}(\beta) = \operatorname{Tors} H^{k+1}(M,\mathbb{Z})$$

But $\operatorname{Im}\psi = H^k(M,\mathbb{R}) / i(H^k(M,\mathbb{Z}))$

= Connected component of $H^k(M,\mathbb{R}/\mathbb{Z})$

Therefore

$$\boxed{\pi_0\left(H^k(M,\mathbb{R}/\mathbb{Z})\right) \cong \mathrm{Tors}\, H^{k+1}(M,\mathbb{Z})}$$

<u>Example:</u> $L_m = S^3/\mathbb{Z}_m$

$$0 \to \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \to H^3(L_m \times L_m, \mathbb{R}/\mathbb{Z}) \to \mathbb{Z}_m \to 0$$

This can be proved using the Künneth theorem to compute $H^*(L_m \times L_m, \mathbb{Z})$ and then the universal coefficient theorem:

$$0 \to H^j(X,\mathbb{Z}) \otimes_{\mathbb{Z}} A \to H^j(X,A) \to \mathrm{Tor}(H^{j+1}(X,\mathbb{Z}),A) \to 0$$

The sequence splits, but not canonically.

First of all, the very definition of a differential character assigns to a character χ a form $F \in \Omega^{\ell}_{\mathbb{Z}}(M)$.

We'll call the map

$$\chi \longmapsto F$$

The "fieldstrength map."

It is a group homomorphism

$$\check{H}^{\ell}(M) \xrightarrow[\text{fieldstr.}]{} \Omega^{\ell}_{\mathbb{Z}}(M) \to 0$$

We will see (e.g. from chain models) that it is surjective.

Example: $l = 1$, $f : M \longrightarrow U(1)$

$$F = -i f^{-1} df$$

Exponential map: $U(1) \cong \mathbb{R}/\mathbb{Z}$,

so $f = e^{i\phi}$. Note!

ϕ not nec. globally well-defined,

we can have $\underbrace{\oint_\gamma d\phi}_{\text{nonzero}} \in \mathbb{Z}$!

Next, we have the topological class

$$\check{H}^{\ell}(M) \xrightarrow[\substack{\text{characteristic} \\ \text{class} \\ X \mapsto c(X)}]{} H^{\ell}(M, \mathbb{Z}) \to 0$$

can show

The general definition of $X \mapsto c(X)$ is best left to the "chain complex" descriptions.

Example: $\ell = 1$:

$$X \to \left[\frac{-i f^{-1} df}{2\pi} \right] \in H^1_{dR, \mathbb{Z}}$$

integral periods

$$\begin{array}{c} H^1_{dR,\mathbb{Z}} \\ \cong \\ H^1(M, \mathbb{Z}) \end{array}$$

Example: $\ell = 2$

Modeling the differential character as the holonomy function of a connection on a principal $U(1)$ line bundle:

$$\chi(\mathcal{W}_1) = \mathrm{Hol}_\nabla(\mathcal{W}_1)$$

for some ∇ on $P \xrightarrow{U(1)} M$

we have

$$c(\chi) = c_1(P) \in H^2(M, \mathbb{Z})$$

This brings up an important new point: As we mentioned M cpt and smooth $\Longrightarrow$ $H^2(M, \mathbb{Z})$ is a fin. gen. Abelian group:

$\therefore$ There is a canonically defined torsion subgroup: s.t.

$$0 \to \mathrm{Tors}(H^2(M,\mathbb{Z})) \to H^2(M,\mathbb{Z})$$

$$\longrightarrow \underbrace{\overline{H}^2(M,\mathbb{Z}) \cong \mathbb{Z}^{b_2}}_{\text{lattice}} \longrightarrow 0$$

DeRham's Theorem:

$$\overline{H}^2(M,\mathbb{R}) = H^2(M,\mathbb{Z}) \otimes \mathbb{R} \cong H^2_{dR}(M)$$

But the torsion subgroup can be nontrivial.

<u>Example:</u> Lens spaces

$$L_m = S^3/\mathbb{Z}_m$$

$$(z_1, z_2) \sim (\omega z_1, \omega z_2) \quad \omega \in \mu_m$$

$$|z_1|^2 + |z_2|^2 = 1$$

$$\pi_1(L_m, x_0) \cong H_1(L_m, \mathbb{Z}) \cong \mathbb{Z}_m$$

$$H^2(L_m, \mathbb{Z}) \cong \mathbb{Z}_m \Rightarrow$$

group of (iso classes of) principal $U(1)$ bundles is just $\mathbb{Z}_m$.

So the characteristic class is entirely torsion.

(Proof: $\pi_1(L_m) \cong \mathbb{Z}_m$ because the action is free and $\pi_1(S^3) = 0$.

$\Rightarrow H_1(L_m) =$ Abelionization of $\pi_1 \cong \mathbb{Z}_m$

Now use the universal coefficient theorem above and $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m$.)

Aside: One can also understand these cohomology groups directly from an equivariant cell-decomposition of S^3:

$\mathbb{Z}_2$-equiv cell decomposition of S^n

$$e_j^{\pm} = \{(x_1, \dots, x_j, x_{j+1}, 0 \dots 0) \in S^n \mid \mathrm{sign}(x_{j+1}) = \pm\}$$

orientation on $e_j^{\pm}$ from $\pm dx^{1 \dots j}$

$$\partial e_j^{+} = e_{j-1}^{+} + e_{j-1}^{-}$$

$$\partial e_j^{-} = -(e_{j-1}^{+} + e_{j-1}^{-}) \quad 1 \leq j \leq n$$

$$\partial e_0^{\pm} = 0$$

Dual cochains: $c_j^{\alpha}(e_k^{\beta}) = \delta^{\alpha\beta}\delta_{jk}$

$\Rightarrow$ Cochain model of $\mathbb{RP}^n = S^n/\mathbb{Z}_2$

$$\mathbb{Z}(c_0^+ - c_0^-) \xrightarrow{0} \mathbb{Z}(c_1^+ + c_1^-) \xrightarrow{2}$$

$$\mathbb{Z}(c_2^+ - c_2^-) \xrightarrow{0} \mathbb{Z}(c_3^+ + c_3^-) \xrightarrow{2} \cdots$$

$\Rightarrow$

$$H^j(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & j = 0 \\ 0 & j \text{ odd}, j < n \\ \mathbb{Z}_2 & j \text{ even } 0 < j < n \\ \mathbb{Z} & j = n, \text{ odd} \\ \mathbb{Z}_2 & j = n \text{ even} \end{cases}$$

$\otimes$ above complex with $\mathbb{Z}_2$ makes all differentials 0 and

$$H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1})$$

Now we return to the exact sequences giving us a picture of the differential cohomology group:

The fieldstrength and topological class maps are beautifully compatible:

$$
\begin{tikzcd}
 & F \in \Omega^\ell_{\mathbb{Z}}(M) \arrow[r] \arrow[dr, "F \mapsto \left[\frac{F}{2\pi}\right]"] & 0 \\
\check{H}^\ell(M) \arrow[ur, "\chi \mapsto F"] \arrow[dr, "\chi \mapsto c(\chi)"'] & & H^\ell_{dR}(M) \\
 & c(\chi) \in H^\ell(M,\mathbb{Z}) \arrow[ur, "\otimes \mathbb{R} + \text{deRham theorem}"'] \arrow[r] & 0
\end{tikzcd}
$$

Now we study the kernels of these homomorphisms

The kernel of $\chi \longmapsto c(\chi)$ are the topologically trivial characters. These are the characters for which $\exists$ globally well-defined $A \in \Omega^{\ell-1}(M)$ s.t.

$$\chi(W_{\ell-1}) = \exp\left(i \int_{W_{\ell-1}} A\right)$$

But remember the character only encodes the gauge invariant information so we identify

$$A \sim A + d\Lambda \quad \text{for } \Lambda \in \Omega^{\ell-2}(M)$$

but even more we could also shift $A \to A + \omega \quad \omega \in \Omega^{\ell-2}_{\mathbb{Z}}(M)$

In physics we make a distinction

$A \to A + d\Lambda$ "small gauge tmn"

"large gauge tmn"

$A \to A + \omega$ if $[\omega] \in H^{\ell-1}_{dR}(M)$ is nonzero

The kernel of $\chi \to F$ are the "flat characters". The subgroup of flat characters are homs $\chi : Z_{\ell-1} \to U(1)$ that only depend on the homology class of W. One can show:

$$\mathrm{Hom}(H_{\ell-1}(M), U(1)) \cong H^{\ell-1}(M, U(1))$$

Now $H^{\ell-1}(M, U(1)) \cong H^{\ell-1}(M, \mathbb{R}/\mathbb{Z})$ is a <u>compact Abelian group</u>

If A is a cpt Ab. group let A_0 = Conn. component of identity.

$$0 \to A_0 \to A \to \underbrace{\pi_0(A)}_{\text{finite Abelian group} \cong \oplus \mathbb{Z}/n_i\mathbb{Z}} \to 0$$

A_0: Connected Abelian group $\cong U(1)^b$

For us: $A = H^{\ell-1}(M, \mathbb{R}/\mathbb{Z})$

$A_0 = H^{\ell-1}(M, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$ "Wilson lines"

Note: Sequence splits, but not canonically (general property of universal coefficient theorem.)

$$\beta : H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \mathrm{Tors}(H^{\ell}(M, \mathbb{Z}))$$

β: Bockstein map: See Bott+Tu.

$$\pi_0\left(H^{\ell-1}(M, \mathbb{R}/\mathbb{Z})\right) \cong \mathrm{Tors}(H^{\ell}(M, \mathbb{Z}))$$

<u>Example</u>: $\ell = 2$, $M = L_m = S^3/\mathbb{Z}_m$

$$H^1(M, \mathbb{R}/\mathbb{Z}) \cong H^2(M, \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$$

$$\pi_1(L_m, x_0) \cong \mathbb{Z}/m\mathbb{Z}$$

Let γ be a generator, we can define <u>flat characters</u> by

$$\chi_r(\gamma) = \exp\left(2\pi i \frac{r}{m}\right)$$

Another good example is:

Putting it all together we have the grand diagram:

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \dfrac{\Omega^{\ell-1}(M)}{\Omega^{\ell-1}_{\mathbb{Z}'}(M)} \arrow[rr, "d"] \arrow[dr] & & \Omega^{\ell}_{\mathbb{Z}'}(M) \arrow[ur] \arrow[dr] & \\
H^{\ell-1}(M,\mathbb{R}) \arrow[ur] \arrow[dr] & & \check{H}^{\ell}(M) \arrow[ur] \arrow[dr] & & H^{\ell}(M,\mathbb{R}) \\
& H^{\ell-1}(M,\mathbb{R}/\mathbb{Z}') \arrow[rr, "\beta"] \arrow[ur] & & H^{\ell}(M,\mathbb{Z}') \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

A dance based on this diagram was choreographed by Kyla Barkin and Aaron Selisson, so we will refer to it as the "dancing hexagon."

Example: $\ell = 1$, $M = S^1$

$$\check{H}^1(S^1) = \mathrm{Map}(S^1 \longrightarrow U(1))$$
$$= \text{loop group } LU(1)$$

Identify domain $S^1 \cong \mathbb{R}/\mathbb{Z}$
with coordinate $\sigma \sim \sigma + n$, $n \in \mathbb{Z}$

oscillator modes

$$f(\sigma) = \exp\Big[i\,\underbrace{\phi_0}_{\substack{\phi_0 \in \mathbb{R}/\mathbb{Z}' \\ \text{flat field}}} + 2\pi i \underbrace{w}_{\substack{w \in \mathbb{Z} \\ \text{characteristic class}}} \sigma + \sum_{n \neq 0} \frac{\phi_n}{n} e^{2\pi i n \sigma} \Big]$$

In general, we have a noncanonical decomposition of the Abelian group $\check{H}^{\ell}(M)$

$$\check{H}^{\ell}(M) \simeq \Gamma_\ell \times T_\ell \times V_\ell$$

Γ_ℓ = discrete group $= H^{\ell}(M, \mathbb{Z})$ topological classes

$$T_\ell = H^{\ell-1}(M, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \approx U(1)^{b_{\ell-1}}$$

torus of Wilson lines

$$V_\ell \cong \mathrm{Im}\left(d^{\dagger}: \Omega^{\ell+1} \longrightarrow \Omega^{\ell}\right)$$

infinite-dim'l vector space of oscillator modes: $d^{\dagger}A = 0$ is a gauge choice

4. Models Of Differential Cohomology

As we have stressed, $\check{H}^{\ell}(M_n)$ is the set of gauge equivalence classes of field configurations of an "$(\ell-1)$-form gauge potential."

When discussing issues where locality is important, such as actions, and gluing principles it is important to have proper models of local gauge potentials.

There are many models available, we'll discuss 2.

Čech version: This formulation goes back to Deligne, and indeed in the context of complex geometry Cheeger-Simons cohomology is also known as Deligne cohomology. It was first introduced into physics by Orlando Alvarez and (independently) by Krysztof Gawedzki:

The basic idea is that on a contractible space all field configurations must be topologically trivial and there is no room for "Wilson lines."

The subtleties arise from patching together local data.

To implement this idea we choose a good cover $\{\mathcal{U}_\alpha\}$ of M. This is a cover s.t. all $\mathcal{U}_{\alpha_1 \cdots \alpha_k} = \mathcal{U}_{\alpha_1} \cap \cdots \cap \mathcal{U}_{\alpha_k}$ are contractible.

We describe the first 3 cases $\ell = 1, 2, 3$:

<u>$\ell = 1$</u>: The periodic scalar

On each $\mathcal{U}_\alpha$ we have $F_\alpha \in \Omega^1(\mathcal{U}_\alpha)$

and
$$F_\alpha = d\phi_\alpha = -i f_\alpha^{-1} d f_\alpha$$

Where $f_\alpha : \mathcal{U}_\alpha \longrightarrow U(1)$ has a well-defined logarithm $\log f_\alpha = i\phi_\alpha$

On $\mathcal{U}_{\alpha\beta}$ $\quad F_\alpha - F_\beta = 0 \implies$

$$f_\alpha^{-1} d f_\alpha - f_\beta^{-1} d f_\beta = 0 \implies$$

$$\implies d\left(f_\alpha / f_\beta\right) = 0 \implies f_\alpha / f_\beta = \text{constant on } \mathcal{U}_{\alpha\beta}$$

Next we impose that the constant is just $f_\alpha / f_\beta = 1$ $\therefore$ f_α patch together to form a well-defined $U(1)$-valued function on M.

Note that $\phi_\alpha - \phi_\beta = 2\pi i\, n_{\alpha\beta}$ on $\mathcal{U}_{\alpha\beta}$

$\therefore$ On $\mathcal{U}_{\alpha\beta\gamma}$:

$$n_{\alpha\beta} + n_{\beta\gamma} + n_{\gamma\alpha} = 0 \quad (*)$$

A collection of integers $n_{\alpha\beta}$ on $\mathcal{U}_{\alpha\beta}$ satisfying $(*)$ is known as a "Čech cocycle"

It is shown in textbooks (e.g. Bott + Tu) that such a cocycle determines a cohomology class in $H^1(M, \mathbb{Z})$.

Remark: If we had just used the functions ϕ_α and only required $\phi_\alpha - \phi_\beta = 2\pi i\, r_{\alpha\beta}$, $r_{\alpha\beta} \in \mathbb{R}$ we would have gotten $H^1(M; \mathbb{R})$

$\underline{l=2}$: $U(1)$ gauge connection

We begin with $F_\alpha \in \Omega^2(\mathcal{U}_\alpha)$

The curvature is globally well-defined

$$F_\alpha - F_\beta = 0 \quad \text{on} \quad \mathcal{U}_{\alpha\beta}$$

On the other hand $dF_\alpha = 0 \Rightarrow$

$F_\alpha = dA_\alpha$ on $\mathcal{U}_\alpha$. (This corresponds to a trivialization of the line bundle on $\mathcal{U}_\alpha$.)

$$dA_\alpha - dA_\beta = 0 \text{ on } \mathcal{U}_{\alpha\beta} \Longrightarrow$$

$$A_\alpha - A_\beta = d\epsilon_{\alpha\beta} = -i\, g_{\alpha\beta}^{-1}\, d g_{\alpha\beta} \quad (*)$$

$$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow U(1) \quad (\text{Note } g_{\beta\alpha} = g_{\alpha\beta}^{-1})$$

$(*) \Rightarrow$ on $\mathcal{U}_{\alpha\beta\gamma}$ $\quad g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = \text{constant}$

We <u>impose</u> the condition that this constant is 1: $\quad g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$ on $\mathcal{U}_{\alpha\beta\gamma}$

Thus the $\{g_{\alpha\beta}\}$ are transition functions for a principal $U(1)$ bundle over M.

Note that on $\mathcal{U}_{\alpha\beta\gamma}$

$$\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma} = n_{\alpha\beta\gamma}$$

$$\Longrightarrow \quad n_{\alpha\beta\gamma} - n_{\alpha\beta\delta} + n_{\alpha\gamma\delta} - n_{\beta\gamma\delta} = 0$$

on $\mathcal{U}_{\alpha\beta\gamma\delta}$ $\Longrightarrow$ $\{n_{\alpha\beta\gamma}\}_{\mathcal{U}_{\alpha\beta\gamma}}$ is a Čech 2-cocycle which ~~and~~

$$\{n_{\alpha\beta\gamma}\} \in H^2_{\check{C}ech}(M, \{\mathcal{U}_\alpha\}) \underset{\text{Bott+TU}}{\cong} H^2(M, \mathbb{Z})$$

The class in $H^2(M, \mathbb{Z})$ is just the first Chern class.

The nice thing about this approach is that it is straightforward to extend it to $l > 2$

$\underline{\ell = 3}$: "gerbe connection"

On $\mathcal{U}_\alpha$ we have local fieldstrength

$$H_\alpha \in \Omega^3(\mathcal{U}_\alpha) \quad \text{with} \quad dH_\alpha = 0$$

$$\Rightarrow \exists \; B_\alpha \in \Omega^2(\mathcal{U}_\alpha) \text{ s.t. } H_\alpha = dB_\alpha$$

$$\Rightarrow \text{On } \mathcal{U}_{\alpha\beta} \quad B_\alpha - B_\beta = d\Lambda_{\alpha\beta} \quad \Lambda_{\alpha\beta} \in \Omega^1(\mathcal{U}_{\alpha\beta})$$

$$\Rightarrow \text{On } \mathcal{U}_{\alpha\beta\gamma} \quad d\left(\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}\right) = 0$$

$$\Rightarrow \text{On } \mathcal{U}_{\alpha\beta\gamma} \quad \Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = -i f_{\alpha\beta\gamma}^{-1} d f_{\alpha\beta\gamma}$$

$$\Rightarrow \text{On } \mathcal{U}_{\alpha\beta\gamma\delta} \quad f_{\alpha\beta\gamma} f_{\alpha\gamma\delta}^{-1} f_{\alpha\gamma\delta} f_{\beta\gamma\delta}^{-1} = \text{constant}$$

Now $\underline{\text{impose}}$ (quantization): On $\mathcal{U}_{\alpha\beta\gamma\delta}$

$$f_{\alpha\beta\gamma} f_{\alpha\gamma\delta}^{-1} f_{\alpha\gamma\delta} f_{\beta\gamma\delta}^{-1} = 1$$

$$\Rightarrow \quad n_{\alpha\beta\gamma\delta} = i \sum \pm \log f \ldots \quad \in 2\pi\mathbb{Z}$$

$\Rightarrow \left\{ \dfrac{n_{\alpha\beta\gamma\delta}}{2\pi} \right\}$ defines an integral class in $H^3(M,\mathbb{Z})$, the topological class of the differential character.

Clearly one can carry this disussion out for any l. Just use more indices and proceed from $F_\alpha \in \Omega^l(\mathcal{U}_\alpha)$ to the degree l Čech class $\{n_{\alpha_1 \cdots \alpha_{l+1}}\}$.

Remark: Gerbe connections arise in several ways in physics. First of all they are used to describe the "Neveu-Schwarz B-field" of string theory.

A second way they arise via lifting conditions from a principal G/Z bundle to a G bundle where G is a connected and simply-connected Lie group and Z is a subgroup of the center.

The transition functions of a principal G/Z bundle satisfy $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$

If one chooses lifts $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ such that $\pi(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}$ then we conclude that

$$\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha} = \zeta_{\alpha\beta\gamma} \quad \text{on } U_{\alpha\beta\gamma}$$

$$\zeta_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \longrightarrow Z$$

For $Z \subset U(1)$ $\{\zeta_{\alpha\beta\gamma}\}$ determine a class in $\check{H}^2(M, Z) \xrightarrow[\beta]{} \check{H}^3(M, \mathbb{Z})$

For example, if we have a bundle $\mathcal{A} \to M$ of algebras with fiber $\mathrm{Mat}_N(\mathbb{C})$ their transition functions will be in $PGL_N(\mathbb{C})$.
The obstruction to identifying
$\mathcal{A} \cong \mathrm{End}(E)$ for some vector bundle $E \to M$

is measured by a gerbe. In this case the class in $H^3(M, \mathbb{Z})$ is called the Dixmier–Douady class.

Applying these ideas to $\pi : Spin(n) \to SO(n)$ leads to the characterization of $w_2(TM)$ as an obstruction to spin structure.
The related bundle of algebras is the bundle of Clifford algebras and the DD class is $W_3(TM) \in H^3(M, \mathbb{Z})$. [Need to double check]

Hopkins-Singer Cocycles: Another local model for "cocycles" corresponding to differential cohomology classes is due to Hopkins + Singer. It is motivated by a "homotopy pushout" construction from homotopy theory. I has two advantages over the Čech description:

1. It can be applied to any generalized cohomology theory. This is especially important in string theory which makes use of "differential K-theory." (See below.)

2. It is easier to view the fields as forming a groupoid. Then, the automorphism group of an object with isom. class in $\check{H}^{\ell}(M)$ is $H^{\ell-2}(M, U(1))$. That in turn is crucial to defining quantum electric charge

One way to motivate the HS model is to consider a formula for the holonomy of a differential character in $\check{H}^\ell$ on an $(\ell-1)$-cycle which is n-torsion.

Thus $\quad n\Sigma = \partial B \quad$ for some integral n-chain B

$$\therefore \quad (\chi(\Sigma))^n = \exp i \int_B F$$

It would be <u>wrong</u> to conclude that

$$\chi(\Sigma) \stackrel{?}{=} \exp \frac{i}{n} \int_B F$$

for one thing $F \in \Omega_{2\pi\mathbb{Z}}(M)$ could have periods $\neq 0 \mod (2\pi n)$. Then the above formula is ill-defined.

However, $\exists\ a \in C^\ell(M,\mathbb{Z})$ s.t.

$$\circledast \quad \chi(\Sigma) = \exp\left[\frac{i}{n}\left(\int_B F - 2\pi\langle a, B\rangle\right)\right]$$

<u>is</u> well-defined.

One can show the class a has $\delta a = 0$ and hence $[a] \in H^\ell(M,\mathbb{Z})$. This is the characteristic class of the character $\chi \in \check{H}^\ell$.

We can write $\circledast$ heuristically as:

$$\delta \log \chi \sim F - a$$

that is the motivating equation for H.S.

<u>Def:</u> A Hopkins–Singer cocycle is a triple

$$x = (a, h, \omega) \in C^\ell(M,\mathbb{Z}) \times C^{\ell-1}(M,\mathbb{R}) \times \Omega^\ell(M)$$

H & S define a <u>cocycle</u> to be a triplex such that

$$\delta h = a_{\mathbb{R}} - \partial h$$

where $a_{\mathbb{R}}$ is the image under $H^{\ell}(M,\mathbb{Z}) \to H^{\ell}(M,\mathbb{R})$ and ω is likewise embedded $\Omega^{\ell}(M) \to H^{\ell}(M,\mathbb{R})$

One could define a chain complex with $\delta^2 = 0$ but it is better (with an eye towards generalizations) to define a groupoid.

The morphism space $\mathrm{Hom}(x,x')$ is

$$\left\{ (b,g) \in \Omega^{\ell-1}(M,\mathbb{Z}) \times \Omega^{\ell-2}(M,\mathbb{R}) \;\middle|\; \begin{array}{l} a - a' = \delta b \\ h - h' = \delta g - b \\ \omega - \omega' = 0 \end{array} \right\} \Big/ \sim$$

The automorphism group is

$$\mathrm{Aut}(x) \cong H^{\ell-2}(M, \mathbb{R}/\mathbb{Z})$$

There are other models described in the book by Amabel, Debray, and Haine.

5. Important Properties Of Differential Cohomology

5A : Ring Structure

There is a graded associative product:

$$\check{H}^{\ell_1} \times \check{H}^{\ell_2} \longrightarrow \check{H}^{\ell_1+\ell_2}$$

$$\chi_1 \cdot \chi_2 = (-1)^{\ell_1 \ell_2} \chi_2 \cdot \chi_1$$

such that:

$$F(\chi_1 \cdot \chi_2) = F(\chi_1) \wedge F(\chi_2)$$

$$c(\chi_1 \cdot \chi_2) = c(\chi_1) \cup c(\chi_2)$$

The formula for the holonomy is more complicated. One way to express it is to give the product in terms of Hopkins-Singer cocycles:

$$(a_1, h_1, \omega_1) \cup (a_2, h_2, \omega_2) :=$$

$$(a_1 \cup a_2,\ \pm a_1 \cup h_2 + h_1 \cup \omega_2 + H(\omega_1, \omega_2),\ \omega_1 \wedge \omega_2)$$

where H is a homtopy betwee cup product $\cup$ and wedge product $\wedge$ on Ω^ℓ, considered as defining (smooth) $\mathbb{R}$-valued ℓ-cochains.

5B: Integration:

If Σ_ℓ is an ℓ-cycle then the holonomy on Σ_ℓ can be considered as an integration

$$\circledast \quad \int_{\Sigma_\ell}^{\check{H}} : \check{H}^{\ell+1}(\Sigma_\ell) \longrightarrow \check{H}^1(pt) = \mathbb{R}/\mathbb{Z}$$

This is a good viewpoint because it generalizes to families.

$$\begin{array}{ccl} M_n & \longrightarrow & \mathcal{X} \quad : \text{Family of } n\text{-manifolds over} \\ & & \downarrow \\ & & \mathcal{S} \quad : \text{control parameters} \end{array}$$

closed n-manifolds

$$\int_{\mathcal{X}/\mathcal{S}}^{\check{H}} : \check{H}^{d}(\mathcal{X}) \longrightarrow \check{H}^{d-n}(\mathcal{S})$$

In the Čech model there is an explicit formula for ⊛ : Choose a triangulation of Σ_ℓ such that ℓ-simplicies sit in a definite $\mathcal{U}_\alpha$ so $\Sigma^{(i)}_{\ell,\alpha} \subset \mathcal{U}_\alpha$ faces $\Sigma^{(i)}_{\ell,\alpha\beta} \subset \mathcal{U}_{\alpha\beta}$ etc. Then denoting the data of the Čech model by

$$\check{A} = (A_\alpha, A_{\alpha\beta}, A_{\alpha\beta\gamma}, \dots)$$

and the corresponding character by

$$\chi = [\check{A}]$$ we have :

$$\exp\left(i \int_{\Sigma_\ell}^{\check{H}} [\check{A}] \right) := $$

$$\prod_{\alpha, j} \exp\left(i \int_{\Sigma^j_{\ell,\alpha}} A_\alpha \right) \cdot \prod_{\alpha<\beta, j} \exp\left(i \int_{\Sigma^j_{\ell,\alpha\beta}} A_{\alpha\beta} \right) \cdots$$

For generalizations to families see Bär + Becker 1303.6457

<u>5C</u>: This leads to a crucial pairing on differential cohomology: multiply and integrate:

2π's?

$$\check{H}^\ell(M_n) \times \check{H}^{n-\ell+1}(M_n) \longrightarrow \check{H}^1(pt) \cong \mathbb{R}/\mathbb{Z}$$

$$\langle [\check{A}_1], [\check{A}_2] \rangle := \int_{M_n}^{\check{H}} [\check{A}_1] \cdot [\check{A}_2] \in \mathbb{R}/\mathbb{Z}$$

There are two important and useful special cases of this pairing:

1.) $[\breve{A}_1]$ is topologically trivial:

$F_1 = dA_1$ for a globally defined $A_1 \in \Omega^{l-1}$

Then: $$\langle [\breve{A}_1], [\breve{A}_2] \rangle = \underbrace{\int_{M_n} A_1 F_2}_{\text{ordinary integral of differential forms}}$$

Note that this implies that if both characters are topologically trivial then the pairing is just

$$\int A_1 dA_2$$

Therefore the pairing can be viewed as one way of generalizing the actions of BF-theory to topologically nontrivial situations. In particular, the 3d Chern-Simons action for Abelian gauge groups can be expressed as a pairing.

2.) If $[\check{A}_1]$ is flat we can regard it as an element $\phi_1 \in H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}')$ (via the hexagon diagram)

$$\exp\left(i\int [\check{A}_1]\cdot[\check{A}_2]\right) = \exp\left(i\int_{M_n} \phi_1 \cup C_2\right)$$

where we use the cup product on $H^{\ell-1}(M, \mathbb{R}/\mathbb{Z}')$ and $H^{n-\ell+1}(M, \mathbb{Z})$ to get an element of $H^n(M, \mathbb{R}/\mathbb{Z}')$.

This observation is important for the discussion of flux sectors below.

5D: The perfect pairing

Using known properties of Pontryagin-Poincaré duality for compact oriented manifolds one can show that

$$\check{H}^{\ell}(M_n) \times \check{H}^{n-\ell+1}(M_n) \longrightarrow \mathbb{R}/\mathbb{Z}$$

is a perfect pairing. That is

$$\mathrm{Hom}\left(\check{H}^{\ell}(M_n), \mathbb{R}/\mathbb{Z}\right) \cong \check{H}^{n-\ell+1}(M_n)$$

This is called Pontryagin–Poincaré duality of differential cohomology. We can use the exact sequences above to provide a proof:

$$\begin{tikzcd}
0 \arrow[r] & H^{\ell-1}(M,\mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[drr, leftrightarrow, "\text{perfect pairing}"] & \check{H}^{\ell}(M) \arrow[r] & \Omega^{\ell}_{\mathbb{Z}}(M) \arrow[r] \arrow[dll, leftrightarrow, "\text{perfect pairing}"'] & 0 \\
0 \arrow[r] & \Omega^{n-\ell}(M)/\Omega^{n-\ell}_{\mathbb{Z}}(M) \arrow[r] & \check{H}^{(n-\ell)+1}(M) \arrow[r] & H^{n-\ell+1}(M,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Remark: There is a nice connection to Dumitrescu's lectures: His $\breve{B}_e^{(2)}, \breve{B}_m^{(2)}$ are external gerbe connections (which couple to 1-form symmetries).

The gerbes can be viewed as external electric and magnetic currents $\check{J}_e, \check{J}_m$ (NOT to be confused with his $J_e^{(2)}, J_m^{(2)}$!)

$$\check{J}_e, \check{J}_m \in \check{H}^3(M).$$

$$\text{Set } \mathcal{S} = \check{H}^3(M) \times \check{H}^3(M)$$

$$\mathcal{X} = \mathcal{S} \times M_4$$

$$\langle \check{J}_e, \check{J}_m \rangle \in \check{H}^2(\mathcal{S})$$

But $\check{H}^2(\mathcal{S})$ is the set of (gauge equiv. classes of) (P, ∇) $P \xrightarrow{U(1)} \mathcal{S}$. The interpretation is that the partition function of Maxwell theory in the presence of simultaneous electric + magnetic current is anomalous and should be viewed as a section of a line bundle given by the class $\langle \check{J}^e, \check{J}^m \rangle \in \check{H}^2(\mathcal{S})$.

Actually, it is a line bundle w/ connection, which surely encodes Ward identities.

6. The Hilbert Space Of A Generalized Abelian Gauge Theory

We now assume a spacetime splitting

$$M_n = N_{n-1} \times \underset{\uparrow \text{time}}{\mathbb{R}}$$

with action : $\displaystyle \pi \int_{M_n} \lambda \, F * F$

(For the periodic scalar in $n=2$ dimensions $\lambda = R^2$ is related to the radius of the target circle.)

The classical momentum is the $(n-\ell)$-form $\Pi = 2\pi\lambda (*_n F)\big|_N$

The classical phase space is

$$T^* \check{H}^\ell(N) = \check{H}^\ell(N) \times \Omega^{n-\ell}(N)/_{\text{Imd}^\dagger}$$

(This follows from the noncanonical decomposition $\check{H}^\ell(N) \approx T_\ell(N) \times \Omega^{\ell-1}(N)/\Omega^{\ell-1}_{\mathbb{Z}}(N)$)

We will use standard quantization so we have Heisenberg relations for the quantum fields F_q, Π_q:

$$\left[\int_{N_{n-1}} \omega_1 F_q , \int_{N_{n-1}} \omega_2 \Pi_q \right] = i\hbar \int_{N_{n-1}} \omega_1 d\omega_2$$

$$\omega_1 \in \Omega^{n-1-\ell}(N_{n-1}), \quad \omega_2 \in \Omega^{\ell-1}(N_{n-1})$$

which is a precise implementation of

$$\Pi \sim -i \frac{\delta}{\delta A}$$

Now we use heavily the property that $\check{H}^{\ell}(N_{n-1})$ is an <u>Abelian group</u>.

Therefore, at least formally, it has a translationally invariant measure so we can formulate the Hilbert space of the theory:

$$\mathcal{H}(N_{n-1}) := L^2(\check{H}^\ell(N_{n-1}))$$

As we have seen $\check{H}^\ell(N_{n-1})$ is an ∞-dim'l group so some analysis is needed to give this formula meaning. From the noncanonical decomposition

$$\check{H}^\ell(N_{n-1}) = \Gamma_\ell \times T_\ell \times V_\ell$$

we see the ∞-dim'l part comes from V_ℓ. The issues here are the same as in the quantization of the free scalar field. The oscillator modes of A with $d^\dagger A = 0$ are quantized as in standard QFT. In these lectures we are more concerned with the subtleties arising from the first two, finite-dimensional, factors. Hence we are somewhat cavalier about the functional-analytic aspects.

Roughly speaking, the allowed wavefunctionals should have Gaussian decay:

$$\Psi[[\check{A}]] \sim P \exp\left(-\int_{N_{n-1}} k F * F \right)$$

where an ON basis for $L^2(\check{H}^\ell(N_{n-1}))$ would involve expressions where P is polynomial in the oscillators from V_ℓ.

17. Some Remarks On Heisenberg Groups

Let A be an Abelian group with Haar measure

$$\tilde{A} := \text{Pontryagin dual}(A) = \text{Hom}(A, U(1))$$

Note that $L^2(A)$ is a representation of A by translation operators

$$(T_{a_0}\psi)(a) := \psi(a+a_0)$$

$$T_{a_0} \circ T_{a_0'} = T_{a_0+a_0'}$$

and it is also a representation of $\tilde{A}$ by multiplication operators:

$$(M_\chi \psi)(a) := \chi(a)\psi(a)$$

$$M_{\chi_1} \circ M_{\chi_2} = M_{\chi_1 \chi_2}$$

but $L^2(A)$ is NOT a representation of $A \times \tilde{A}$ because:

$$T_{a_0} M_\chi = \chi(a_0)\, M_\chi T_{a_0}$$

Rather, $L^2(A)$ is a representation of the Heisenberg extension which, as a <u>set</u> is $U(1)\times A\times\tilde{A}$ but has a group law:

$$\big(z_1, (a_1,\chi_1)\big)\cdot\big(z_2,(a_2,\chi_2)\big) :=$$

$$\big(z_1 z_2 \chi_1(a_2), (a_1+a_2, \chi_1\chi_2)\big)$$

Therefore the Heisenberg group sits in an exact sequence:

$$1 \to U(1) \to \mathrm{Heis}(A\times\tilde{A}) \to A\times\tilde{A} \to 1$$

Now the key theorem on representations of $\mathrm{Heis}(A\times\tilde{A})$ is:

<u>Theorem</u> [Stone-von Neumann-Mackey]
Up to isomophism there is a <u>unique</u>, unitary irrep of $\mathrm{Heis}(A\times\tilde{A})$ such that the central $U(1)$ acts by scalars.

The proof can be found in many places. One of them (with further references) is my group theory notes section 15.5.5.

One model for the SvN representation is $L^2(A)$ with A and $\tilde{A}$ acting as translation + multiplication operators.

If A is locally compact then Pontryagin duality says $\widetilde{(\tilde{A})} \cong A$

Therefore another equivalent representation is $L^2(\tilde{A})$ where $\tilde{A}$ acts by translation and A acts by multiplication. The isomorphism $L^2(A) \cong L^2(\tilde{A})$ is Fourier transformation.

8. Manifest Electro-Magnetic Duality

We can now apply the remarks of sec. 7 to the GAGT. We observed that the Hilbert space should be $L^2(\check{H}^{\ell}(N_{n-1}))$.

But Pontryagin-Poincaré duality says that this is the unique Stone-von Neumann rep$^{\underline{n}}$ of

$$\mathrm{Heis}\left(\check{H}^{\ell}(N_{n-1}) \times \check{H}^{n-\ell}(N_{n-1})\right)$$

We could switch the factors and equally well say it is

$$L^2\left(\check{H}^{n-\ell}(N_{n-1})\right)$$

Thus, Abelian S-duality is nothing but Fourier transformation.

9. The Definition — And Noncommutativity — Of Quantum Electric + Magnetic Fluxes

We have seen that the periods of the fieldstrength of a differential character are quantized

$$\int_{\Sigma_\ell} F \in 2\pi \mathbb{Z} \quad \text{for } \partial\Sigma_\ell = 0$$

However $\int_{\Sigma_{n-\ell}} *F$ is definitely

not quantized! We could continuously change the metric and alter the result. There is thus some tension with electromagnetic duality. We can resolve this puzzle by thinking more carefully about the quantum definition of flux.

As we have noted, if $M_n = N_{n-1} \times \mathbb{R}$ and $\mathcal{H} = L^2(\check{H}^\ell(N_{n-1}))$ then the generator of translations is

$$\Pi_{\check{\phi}} = \lambda(*F_{\check{\phi}})\Big|_{N_{n-1}}$$

So a translation eigenstate would satisfy

$$\Psi(\check{A} + \check{\phi}) = \exp\left(2\pi i \int_{N_{n-1}}^{\check{H}} \check{\mathcal{E}} \cdot \check{\phi}\right) \Psi(\check{A})$$

It is natural to regard the eigenvalue

$\check{\mathcal{E}} \in \check{H}^{n-\ell}(N_{n-1})$ as the quantum definition of definite "electric flux" (we will soon alter the meaning of this term)

Eigenstates of the translation operator are plane waves on fieldspace. They are definitely not L^2 so this notion of electric flux is of limited utility.

A more useful definition of electric flux is the topological class of $\check{\mathcal{E}}$, a class in $H^{n-\ell}(N_{n-1}, \mathbb{Z})$.

We note that $\check{\mathcal{E}}_1, \check{\mathcal{E}}_2$ are in the same path component of $\check{H}^{n-\ell}(N_{n-1})$ iff for all <u>flat fields</u> $\phi_f \in H^{\ell-1}(N, \mathbb{R}/\mathbb{Z})$ we have

$$\int_N^{\check{H}} \check{\phi}_f \cdot \check{\mathcal{E}}_1 = \int_N^{\check{H}} \check{\phi}_f \cdot \check{\mathcal{E}}_2$$

This leads us to the crucial definition:

Def: A state $\Psi \in L^2(\check{H}^\ell(N))$ is in a state of definite electric flux, if it is a translation eigenstate under the subgroup of <u>flat</u> <u>fields</u>.

$\forall\ \phi_f \in H^{\ell-1}(N, \mathbb{R}/\mathbb{Z})$

$$\Psi(\check{A} + \hat{\phi}_f) = \exp\left(2\pi i \int_N e\,\phi_f\right)\Psi(\check{A})$$

for some $e \in H^{n-\ell}(N, \mathbb{Z})$.

e is the quantized electric flux and is the proper quantization of $[*F]$.

<u>Remark:</u> In modern language, if the Hamiltonian dynamics is invariant under the shift by a flat field we say there is an "$(\ell-1)$-form symmetry." The "$(\ell-1)$-form symmetry group" is $H^{\ell-1}(Y, \mathbb{R}/\mathbb{Z})$

Our definition of electric flux is that it is a character of the "$(\ell-1)$-form symmetry group."

Now recall that $\check{H}^\ell(N)$ has connected components

$$\check{H}^\ell(N) = \coprod_{m \in H^\ell(N,\mathbb{Z})} \check{H}^\ell(N)_m$$

Def: A state of definite magnetic flux $m \in H^\ell(N,\mathbb{Z})$ is a wavefunction ψ with support in the component $\check{H}^\ell(N)_m$.

Let $\mathcal{U}_e(\phi_f^e)$ $\qquad \phi_f^e \in H^{\ell-1}(N,\mathbb{R}/\mathbb{Z})$ be the translation operator by flat fields.

Of course, $L^2(\check{H}^\ell(N)) \cong L^2(\check{H}^{n-\ell}(N))$

So, there is a corresponding operator

$\mathcal{U}(\phi_f^m) \qquad \phi_f^m \in H^{n-\ell-1}(N, \mathbb{R}/\mathbb{Z})$

of translation by flat magnetic fields in $L^2(\check{H}^{n-\ell}(N.))$. In the SvN representation $L^2(\check{H}^{\ell}(N))$, $\mathcal{U}_m(\phi_f^m)$ acts as a multiplication operator so:

A state of definite magnetic flux is an eigenstate of $\mathcal{U}(\phi_f^m)$ for all flat magnetic fields $\phi_f^m \in H^{n-\ell-1}(N, \mathbb{R}/\mathbb{Z})$

Thus our definitions of quantum electric and magnetic fluxes are duality invt.

Now, recall that the compact Abelian group $H^{\ell-1}(N, \mathbb{R}/\mathbb{Z})$ might be disconnected.

$$\pi_0\left(H^{\ell-1}(N,\mathbb{R}/\mathbb{Z})\right) = \mathrm{Tors}\left(H^{\ell}(N,\mathbb{Z})\right)$$

When this group of components is nontrivial

It is impossible to have an eigenstate of translations by all flat fields ϕ_f^e which is supported in a single component!

In other words, in general electric + magnetic fluxes cannot be simultaneously measured.

It is not difficult to show that

$$\mathcal{U}_e(\phi_f^e)\,\mathcal{U}_m(\phi_f^m) = e^{2\pi i\, T(\phi_f^e,\phi_f^m)}\,\mathcal{U}_m(\phi_f^m)\,\mathcal{U}_e(\phi_f^e)$$

where T is the torsion pairing

$$T: H^{\ell-1}(N,\mathbb{R}/\mathbb{Z}) \times H^{n-\ell-1}(N,\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z}$$

Remark: In currently fashionable language one way of phrasing this result is: "There is an 't Hooft anomaly between electric and magnetic $(\ell-1)$-form and $(n-\ell-1)$-form symmetries."

Example: A good example is Maxwell theory $(\ell=2)$ on a Lens space $S^3/\mathbb{Z}_m$ where

$$H^1(L_m, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_m$$

In this case the electric and magnetic translation operators generate a Heisenberg group

$$0 \to \mathbb{Z}_m \to \mathrm{Heis}(\mathbb{Z}_m \times \mathbb{Z}_m) \to \mathbb{Z}_m \times \mathbb{Z}_m \to 0$$

The translation by flat fields is clearly a symmetry of the

Hamiltonian, so we conclude that the groundstate of the Maxwell theory must be degenerate: It must contain at least one copy of the m-dim'l irrep of Heis($\mathbb{Z}_m \times \mathbb{Z}_m$).

One might wonder if one could use this result to get a topoligical Qbit (or Qdit). An attempt was made in 0706.3410, but it has received little attention.

A quite notable feature of this phenomenon is that it is a <u>long-distance</u>, <u>macroscopic</u>, quantum phenomenon.

10. Generalized Cohomology Theories And Their Differential Cohomology

10A: The Eilenberg–Steenrod axioms

Cohomology is a functor from pairs of topological spaces (X, A) $A \subset X$ to $\mathbb{Z}$-graded Abelian groups: $(X,A) \longrightarrow \bigoplus_k H^k(X,A)$ such that $\exists\ \delta^*: H^k(A) \longrightarrow H^{k+1}(X,A)$ such that:

1) Homotopy invariance
2) LES $i: A \hookrightarrow X$ $\quad j: (X, \phi) \hookrightarrow (X,A)$

$$\cdots \longrightarrow H^k(X,A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta^*} H^{k+1}(X,A) \longrightarrow \cdots$$

3.) Excision: $\mathrm{Int}(U) \subset \mathrm{Int}(A) \Longrightarrow$

$$H^k(X,A) \approx H^k(X-U, A-U)$$

4.) Additivity:

$$H^k\left(\coprod_\alpha X_\alpha\right) \cong \bigoplus_\alpha H^k(X_\alpha)$$

5.) Point axiom:

$$H^k(pt, \phi) = \begin{cases} \mathbb{Z} & k=0 \\ 0 & k\neq 0 \end{cases}$$

(1) – (4) $\Rightarrow$ Mayer–Vietoris then together with (5) can compute.

Generalized Cohomology Theories (a.k.a. extraordinary cohomology theories) Satisfy axioms (1) $\to$ (4) but replace the point axiom.

e.g. (complex) K theory has $K^j(pt) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$

Hopkins + Singer showed that to any G.C.T. there is a differential version, that satisfies an analog of the hexagon diagram.

Differential K-theory (in various versions is thought to be the proper description of the gauge-invariant information in the RR field)

11. A Hilbert Space For The Self-Dual Field.

We can apply these ideas to the self-dual field in $n = 2 \bmod 4$ dimensions with classical equations of motion

$$F = *F \quad \& \quad dF = 0$$

$$F \in \Omega^{\ell}(M_n) \qquad n = 2\ell$$

$$\therefore \quad \ell = 2s+1 \quad n = 4s+2$$

The nonself-dual field provides a SvN representation of $\mathrm{Heis}(\check{H}^{\ell} \times \check{H}^{\ell})$

We would like to take 1/2 the degrees of freedom.

The idea is to make sense of $Heis(\check{H}^{\ell}(N))$ by viewing $\check{H}^{\ell}(N)$ as a group with symplectic form.

We need another Theorem from group theory:

<u>Thm:</u> Let G be a topological Abélian group. The isomorphism classes of central extensions

$$1 \longrightarrow U(1) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

are in 1-1 correspondence with continuous, alternating, bihomomorphisms

$$s : G \times G \longrightarrow U(1)$$

Alternating means $s(x,x) = 1$ and implies skew $s(x,y) = s(y,x)^{-1}$.

Here $s(x,y)$ is the commutator function: If

$$(z_1, x_1) \cdot (z_2, x_2) = (z_1 z_2 c(x_1, x_2), x_1 + x_2)$$

is a central extension

$$s(x,y) = \frac{c(x,y)}{c(y,x)}$$

The point is: given the commutator

function one can deduce the central extension up to isomorphism.

So to find

$$1 \to U(1) \to \mathrm{Heis}(\check{H}^{\ell}(N)) \to \check{H}^{\ell}(N) \to 1$$

it will suffice to find a suitable s-function on $\check{H}^{\ell}(N)$.

There is a canonical choice given by the pairing:

$$s_{\text{trial}}(x_1, x_2) = \exp(i \langle x_1, x_2 \rangle)$$

This is certainly a bihomomorphism. Because ℓ is odd it is skew, but it is not alternating. Rather one can show:

$$S_{trial}(\mathcal{X},\mathcal{X}) = (-1)^{\int_N \nu_{2s} \cup c(\mathcal{X})}$$

ν_{2s} = degree $2s$ Wu class of N. For oriented mflds...

$$\nu_0 = 1 \quad \nu_2 = w_2 \quad \nu_4 = w_4 + w_2^2$$

$$c(\mathcal{X}) \in H^\ell(N,\mathbb{Z}) = H^{2s+1}(N,\mathbb{Z})$$

We therefore introduce a

$\mathbb{Z}_2$-grading of $\check{H}^\ell(N)$

$$\epsilon(\mathcal{X}) = \begin{cases} 0 & \int \nu_{2s} \cup c(\mathcal{X}) = 0 \bmod 2 \\ 1 & \int \nu_{2s} \cup c(\mathcal{X}) = 1 \bmod 2 \end{cases}$$

$$s(x, x') = \exp\left(i \langle x, x' \rangle - i\pi \epsilon(x)\epsilon(x')\right)$$

is now a bihomomorphism which is alternating, so

$\mathrm{Heis}(\check{H}^{\ell}(N))$ is a $\mathbb{Z}_2$-graded Heisenberg group.

There is a <u>unique</u> $\mathbb{Z}_2$-graded uniary SvN irrep: This is the Hilbert space of the self-dual field (up to isom.)

Very similar remarks apply to the RR field of type $\mathbb{II}$ string theory.

A curious consequence:

The Hilbert space is naturally $\mathbb{Z}_2$-graded. We can indeed interpret this as a boson/fermion grading. This should not be terribly surprising since the chiral scalar in $\mathbb{M}^{1,1}$ is related to the chiral fermion. If we KK reduce a self-dual field on $\mathbb{M}^{1,1} \times Y$, $\dim Y = 4s$, then we get a theory of chiral and antichiral scalars. The fermionic parity of vertex operators in this $1+1$ theory is related to the parity of x^2 where $x \in H^{2s}(Y, \mathbb{Z})$.

There will be similar phenomena with the RR field of type II strings.

12. Applications To The M-Theory Abelian Gauge Field.

The bosonic fields of 11d sugra:

$g_{\mu\nu} \in MET(M_{11})$ and C, a "3-form gauge potential."

When $C \in \Omega^3(M_{11})$ it has action (Lor. signature)

$$\exp\left(i\pi \int_{M_{11}} \lambda\, G \wedge * G + 2\pi i \int_{M_{11}} \left(\frac{1}{6} CGG - C\, I_8(g) \right) \right)$$

$I_8(g)$: Chern-Weil representative of

$$\frac{4p_2 - p_1^2}{4 \cdot 48} \qquad G = dC$$

Various arguments suggest the gauge-invariant info in the C-field is a character $\chi \in \check{H}^4(M_{11})$. This is not quite the case: Witten showed that cancellation of anomalies on the M2 brane $\Rightarrow$

$$\int_{\Sigma_4} \left(G - \frac{1}{4} p_1(TM) \right) \in \mathbb{Z}$$

So a better description is that the gauge invariant info of the C-field is in a <u>torsor</u> for $\check{H}^4(M_{11})$. (The C-field is a differential chain that trivializes a backgnd magnetic current in $\check{H}^5(M_{11})$ related to ω_4.)

Due to some mathematical "coincidences" one can give (see Diaconescu–Freed–Moore for details) a groupoid modeling the gauge-variant information describing the C-field. We have triples (P, ∇, c) where $P \longrightarrow M_{11}$ is a principal E_8-bundle with connection ∇ and $c \in \Omega^3(M_{11})$ is a globally defined 3-form.

$(P, \nabla, c) \sim (P', \nabla', c')$ if

$$c' - c = CS(\nabla, \nabla')$$

and we identify the fieldstrength as:

$$G = \mathrm{tr}\, F^2 - \frac{1}{2} \mathrm{tr}\, R^2 + dc$$

Let $i(\not{D})$ = index density of Atiyah & Singer.

Witten observed the following string miracle:

$$\Big[\frac{1}{2} i(\not{D}_{P,\nabla}) + \frac{1}{4} \underbrace{i(\not{D}_{RS})}_{\text{Rarita-Schwinger field}} \Big]^{(12)} = \cdots$$

$$= \frac{1}{6} G^3 - G I_8 + d(\underbrace{z_{local}}_{\substack{\text{globalley} \\ \text{defined} \\ \text{11-form.}}})$$

$\Longrightarrow$ via APS index theorem we can give an intrinsic 11d definition of the M-theory phase:

$$\Phi(X) = \exp\left\{ 2\pi i \left(\frac{\xi(D\!\!\!/_{\,?,\nabla})}{2} + \frac{\xi(D\!\!\!/_{RS})}{4} \right) + 2\pi i\, I_{8,local} \right\}$$

$$\xi(D\!\!\!/) := \frac{1}{2}\left(\eta(D\!\!\!/) + h(D\!\!\!/) \right)$$

$$I_{loc} = \int \frac{1}{2} c G^2 - \frac{1}{2} c\, dc\, G + \frac{1}{6} c (dc)^2 - c I_8$$

Witten, and Freed-Moore (Setting....)

Show that

$$\mathrm{Pfaff}(\not{D}_{RS})\,\Phi(X)$$

is globally well-defined $\boxed{\text{YK}}$